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Bifurcation and Stability of Radially Symmetric Equilibria of a Parabolic Equation with Variable Diffusion

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1. INTRODUCTION

The purpose of this paper is to give the global bifurcation diagram and the stability of the radially symmetric equilibrium solution of the problem

$$\frac{\partial u}{\partial t} = \nabla \cdot [a^2(x) \nabla u] + \lambda f(u), \qquad (t, x) \in (0, \infty) \times D$$

$$u = 0 \qquad \text{on } \partial D, \tag{1.1}$$

where $\lambda \in \mathbb{R}^+$ is the bifurcation parameter, $D = \{x \in \mathbb{R}^N \colon \|x\| < 1\}$, $\|\cdot\|$ is the Euclidean norm, $a(x) > 0$ is a radially symmetric function in $C^2(\bar{D})$, $u(0, \cdot) \in H_0^1(D) = W_0^{1,2}(D)$, and f is a C^2-function satisfying

$$f(0) = 0, \qquad f'(0) > 0 \tag{1.i}$$

$$\operatorname{sgn} f''(u) = -\operatorname{sgn} u, \qquad \forall u \in \mathbb{R},\ u \neq 0 \tag{1.ii}$$

$$\limsup_{\|u\| \to \infty} f(u)/u \leqslant 0. \tag{1.iii}$$

Equivalently, the bifurcation parameter λ could have been considered as the radius of the ball D instead of the formulation above.

The above problem is equivalent to the following one:

$$(a^2(r)\, u_r)_r + (N-1)\, a^2(r)\, \frac{u_r}{r} + \lambda f(u) = 0, \qquad 0 < r < 1 \tag{1.2}$$

$$u_r(0) = 0, \qquad u(1) = 0.$$

System (1.1) defines a dynamical system in $H_0^1(D)$ (see [4]). From the theory of regularity of solution of elliptic equations, the equilibrium

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solutions of (1.1) are smooth enough to justify all the manipulations carried out hereafter. The smoothness condition requires that $a_r(0)=0$.

The case of a Neumann boundary condition, namely $\nabla u \cdot \hat{n}=0$ on ∂D, where $\hat{n}$ is the unit vector orthogonal to ∂D, will also be discussed. In this case condition $u(1)=0$ in (1.2) should be replaced by $u_r(1)=0$.

To summarize our result, let λ_n, $n \geqslant 0$, be the $(n+1)$th eigenvalue of the boundary value problems:

$$(a^2(r)\,u_r)_r+(N-1)\,a^2(r)\,\frac{u_r}{r}+\lambda f'(0)u=0, \qquad r\in(0,1)$$

$$u_r(0)=0, \qquad u(1)=0. \tag{1.3}$$

Then our main result can be stated as follows.

THEOREM 1.1. *Suppose f satisfies* (1.i), (1.ii), (1.iii) *and the diffusion function $r^2a_{rr}+(N-1)\,ra_r \leqslant (N-1)a$ for $0<r<1$. If $\lambda\in(\lambda_{n-1},\lambda_n)$, $n\geqslant 1$, then there are exactly $2n+1$ radially symmetric equilibrium solutions of* (1.1)*. For $0<\lambda<\lambda_0$, the only equilibrium solution is the zero one. For $\lambda>\lambda_0$, the only stable equilibria are the ones with no zeros in $0<r<1$. If C_k, $k\geqslant 0$, denotes the branch of solutions of* (1.2) *emanating from the zero solution at $\lambda=\lambda_k$, then C_k is an unbounded continuum and the solutions in it are characterized by having exactly k simple zeros in $0<r<1$, for all $\lambda>\lambda_k$.*

Theorem 1.1 extends our previous paper [1] to the case of variable diffusion. For the one-dimensional case, i.e., $N=1$, the condition on the diffusion function $a(r)$, so that (1.1) does not possess a nonconstant stable equilibrium solution, in the case of a Neumann boundary condition and a stable equilibrium solution which vanishes somewhere in (0, 1), in the case of a Dirichlet boundary condition, becomes $a_{rr}<0$.

Therefore, it extends results of Yanagida [2] and Hale and Chipot [3] to radially symmetric equilibria of (1.1) in a ball. See Remark 3 at the end of this paper.

Problem (1.1) can also be viewed as a generalization to n-dimensions ($n=2,3$) of the selection–migration model considered by Fife and Peletier [5], in connection with the study of clines after an appropriate rescaling of the spatial variable and taking $s(x)=\lambda>0$, the intensity of selection as the bifurcation parameter.

The method of proof is to first show that local bifurcations from the zero solution occur as expected at the eigenvalues λ_n. This is accomplished by an application of the theorem on bifurcation from a simple eigenvalue. By using comparison techniques based on maximum principles, it is proved that zero is not an eigenvalue of the linearized equation around any radially symmetric equilibrium. This implies that the bifurcating branches present no secondary bifurcation and are monotone in λ.

As for stability, we prove that an equilibrium solution ϕ, of the problem

$$\begin{gathered}\frac{\partial u}{\partial t}=(a^2(r)\,u_r)_r+(N-1)\,a^2(r)\,\frac{u_r}{r}+\lambda f(u)\\ u_r(0)=0, \qquad u(1)=1,\end{gathered} \tag{1.4}$$

is an unstable solution of (1.4) and of (1.1) if ϕ vanishes somewhere in $(0, 1)$, provided that the function $a^2(r)$ satisfies $r^2 a_{rr}+(N-1)\,r a_r \leqslant (N-1)a$. In the notation of Theorem 1.1, it means that C_0 is the only branch of stable equilibria. In the case of the Neumann boundary condition, if $f(a)=f(b)=0$ then $C_0=\{\phi=a\}\cup\{\phi=b\}$.

Whenever convenient and the context is clear we will consider solutions of (1.4) as solutions of (1.1) without further comments.

2. Local Bifurcation Diagram

We start this section with a result concerning the following eigenvalue problem:

$$\begin{gathered}\nabla[a^2(\|x\|)\,\nabla u]+\lambda f'(0)u=0\\ u|_{\partial D}=0.\end{gathered} \tag{2.1}$$

Let us take, for simplicity, $f'(0)=1$.

Lemma 2.1. *Suppose* $\lambda_0'=\inf\{\int_D a^2(x)\,|\nabla u|^2\,dx,\ \int_D u^2=1\}$. *Then* λ_0' *is the first eigenvalue of* (2.1). *Moreover it is a simple eigenvalue with a corresponding eigenfunction* $u_0(x)>0$ *in* D.

Proof. That λ_0' is the first eigenvalue of (2.1) follows from the variational characterization of eigenvalues. If u_0 minimizes the quadratic form, so does $|u_0|$ and therefore we may assume $u_0>0$ in D.

In order to prove that λ_0' is a simple eigenvalue, i.e., that the eigenspace corresponding to λ_0' is one-dimensional, we suppose that ψ also satisfies (2.1) and prove that u_0 differs from ψ by a multiplicative constant.

By combining the equations we obtain:

$$\begin{aligned}0&=u_0\nabla(a^2\,\nabla\psi)-\psi\nabla(a^2\,\nabla u_0)=\nabla(u_0 a^2\,\nabla\psi)-\nabla(\psi a^2\,\nabla u_0)\\ &=\nabla[a^2(u_0\,\nabla\psi-\psi\,\nabla u_0)]=\nabla\left[a^2u_0^2\nabla\left(\frac{\psi}{u_0}\right)\right]=0, \qquad \text{in } D.\end{aligned}$$

Integrating by parts it follows that:

$$
\begin{aligned}
0 &= \int_D \frac{\psi}{u_0} \nabla\left[a^2 u_0^2 \nabla\left(\frac{\psi}{u_0}\right)\right] dx = \int_D a^2 u_0 \psi \Delta\left(\frac{\psi}{u_0}\right) dx \\
&\quad + \int_D \frac{\psi}{u_0} \nabla(a^2 u_0^2) \nabla\left(\frac{\psi}{u_0}\right) dx \\
&= \int_D a^2 u_0 \psi \Delta\left(\frac{\psi}{u_0}\right) dx - \int_D a^2 u_0^2 \nabla\left[\frac{\psi}{u_0} \nabla\left(\frac{\psi}{u_0}\right)\right] dx \\
&= \int_D a^2 u_0 \psi \Delta\left(\frac{\psi}{u_0}\right) dx - \int_D a^2 u_0^2 \left|\nabla\left(\frac{\psi}{u_0}\right)\right|^2 dx \\
&\quad - \int_D a^2 u_0 \psi \Delta\left(\frac{\psi}{u_0}\right) dx \\
&= -\int_D a^2 u_0^2 \left|\nabla\left(\frac{\psi}{u_0}\right)\right|^2 dx.
\end{aligned}
$$

To use Green's Theorem we define ψ/u_0 and each component of $\nabla(\psi/u_0)$ as well as its derivatives on ∂D, using a limit process, so as to make it functions of $H_1(D)$. Therefore our claim follows.

LEMMA 2.2. *Let λ_0 be the first eigenvalue of problem* (1.3). *Then $\lambda_0' = \lambda_0$, i.e., the eigenfunction referred to in Lemma* 2.1, *is a radially symmetric function in $\bar{D}$.*

Proof. For the sake of simplicity we render the proof just for the case $N = 2$.

By introducing polar coordinates, problem (2.1), with $\lambda_0 = \lambda_0'$, is reduced to the following one:

$$
\begin{aligned}
&(a^2 u_{o,r})_r + a^2 \frac{u_{o,r}}{r} + \frac{a^2 u_{o,\theta\theta}}{r^2} + \frac{(a^2)_\theta u_{o,\theta}}{r^2} + \lambda_0 f'(0)\, u_o = 0, \qquad 0 < r < 1 \\
&\qquad\qquad 0 \leqslant \theta < 2\pi \qquad\qquad (2.2) \\
&\qquad u_0(1, \theta) = 0, \qquad \text{for } 0 \leqslant \theta < 2\pi.
\end{aligned}
$$

In addition, we know that dim kernel

$$[\nabla(a^2 \nabla \cdot) + \lambda_0 f'(0) \cdot] = 1$$

and

$$\int_0^1 \int_0^{2\pi} u_0^2(r, \theta) r \, d\theta \, dr = 1.$$

It is easy to see that for any $\theta_0 > 0$, $u_0(r, \theta + \theta_0)$ is also a solution of (2.2). Therefore, since λ_0 is a simple eigenvalue, there exist a constant k such that $u_0(r, \theta) = ku_0(r, \theta_0 + \theta)$.

In order to have $a^2(r, \theta)$ a single-value function we must suppose u_0 is 2π-periodic in θ. This implies

$$\int_0^1 \int_0^{2\pi} u_0^2(r, \theta_0 + \theta)\, d\theta\, dr = \int_0^1 \int_{\theta_0}^{2\pi + \theta_0} u_0^2(r, \phi) r\, d\theta\, dr$$

$$= \int_0^1 \int_0^{2\pi} u_0^2(r, \phi) r\, d\theta\, dr = 1.$$

Hence $k = \pm 1$, i.e., $u_0(r, \theta) = \pm u_0(r, \theta_0 + \theta)$ for any $\theta_0 > 0$, $0 \leqslant r \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$ and Lemma 2.2 is proved.

The next lemma can be proved by extending classical results on singular Sturm–Liouville problems to the following eigenvalue problem:

$$\begin{gathered}(a^2(r)\, u_r)_r + (N-1)\, a^2(r)\, \frac{u_r}{r} + \lambda f'(0) u = 0, \qquad r \in (0, 1)\\ u_r(0) = 0, \qquad u(1) = 0.\end{gathered} \tag{2.3}$$

Let us take $f'(0) = 1$, for simplicity.

LEMMA 2.3. *The eigenvalues of the singular Sturm–Liouville problem* (2.3) *form an increasing sequence* $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$ *such that* $\lim_{n \to \infty} \lambda_n = \infty$. *There is a unique eigenfunction* u_n, $n \geqslant 0$ (*except for a multiplicative constant*) *associated to the simple eigenvalue* λ_n *and* u_n *has* n *simple zeros in* (0, 1).

We have seen in Lemma 2.2 that $\lambda_0 = \lambda_0'$.

It is worthwhile to remark that when $a^2(x) = 1$, $N \geqslant 2$, $f'(0) = 1$, then $\lambda_n = \alpha_n^2$, where the α_n's are the positive zeros of the Bessel function $J_{(N-2)/2}$ with corresponding eigenfunctions given by $u_n(r) = r^{-(N-2)/2} J_{(N-2)/2}(\alpha_n r)$.

Let us just sketch the proof that the eigenvalue λ_n's are simple.

To this end we call $H_r(D)$ and $L_r^2(D)$, to the functions of $H_0^1(D) \cap H^2(D)$ and $L^2(D)$, respectively, which are radially symmetric on D.

We remark without going through the computations that problem (2.3) is equivalent to the integral equation

$$u(r) = u(0) + \lambda \int_0^r s^{N-1} \int_r^s \frac{dt}{t^{N-1} a^2(t)}\, u(s)\, ds, \qquad 0 < r < 1, \tag{2.4}$$

where the kernel $K(r, s) = s^{N-1} \int_r^s dt/t^{N-1} a^2(t)$ satisfies $\int_0^1 \int_0^1 |K(r, s)|^2\, dr\, ds < \infty$.

Suppose now that for a fixed eigenvalue λ_k, $k>0$, there are two solutions of (2.4), say, u_1 and u_2, with $u_1(0)\neq 0$ and $u_2(0)\neq 0$. Consider the following solution of (2.4): $v(r)=cu_1(r)$ where $c=u_2(0)/u_1(0)$. Then the function $w(r)=v(r)-u_2(r)$ satisfies $w(r)=\lambda_k \int_0^r s^{N-1} \int_r^s (dt/t^{N-1}a^2(t))\,w(s)\,ds$ and $w(0)=0$.

Resorting now to the uniqueness of the solution of this kind of equation in $L_r^2[0,1]$, we conclude that $u_2(r)=cu_1(r)$, $0\leqslant r\leqslant 1$. This assures us that λ_k is a simple eigenvalue.

The operator $A(\lambda,u)=\nabla[a^2(\|x\|)\,\nabla u]+\lambda f(u)$ acting from $H_r(D)$ into $L_r^2(D)$ is in $C^2(\mathbb{R}^+\times H_r(D), L_r^2(D))$ if $H_r(D)$ is equipped with the usual norm of $H^2(D)$.

Furthermore, $A(\lambda,0)=0$, $\forall\lambda\in\mathbb{R}^+$, and the linearized operator $A_u(\lambda,0)$ is self-adjoint.

Now it is standard procedure to check that the hypotheses of the so-called "bifurcation from a simple eigenvalue theorem" are satisfied, thus yielding

LEMMA 2.4. *If λ_n is an eigenvalue of* (2.3) *with corresponding eigenfunction ϕ_n, then $(\lambda_n,0)$ is a bifurcation point for solutions of $A(\lambda,u)=0$, with respect to the curve $\{(\lambda,0), \lambda\in\mathbb{R}^+, 0\in H_r(D)\}$ and there is a unique C^1-curve of solutions $(\lambda(s),u(s))=(\lambda(s), s\phi_n+v(s))$ for $|s|$: small, such that $(\lambda(0),u(0))=(\lambda_n,0)$. Moreover locally these bifurcating curves consist of radially symmetric equilibrium solutions of* (1.1).

More information can be derived about the behavior of the bifurcating branch of $A(\lambda,u)=0$ near a bifurcation point $(\lambda_n,0)$.

By supposing further that $f'''(0)<0$, well-known computations show that the bifurcation curves branch to the right. To illustrate we suppose that this is the case throughout this paper.

Following notation set forth in Theorem 1.1, in a neighborhood of $(\lambda_k,0)$ the equilibrium solutions of (1.4) in C_k denoted by ϕ_k are characterized by having k simples zeros in (0, 1).

The number of zeros follows from the fact that in this neighborhood the equilibrium ϕ_k of (1.4) can be written as $\phi_k(r)=su_k(r)+v(s,r)$, where $v(0,r)=(d/ds)\,v(0,r)=0$ and u_k has k zeros in (0, 1).

That these zeros are simple follows from an argument based on the uniqueness of the solutions of ordinary differential equations.

Later on this feature of the equilibria in C_k will be proved to hold globally and there will be only one equilibrium having a fixed number of maxima and minima.

3. Continuation of Bifurcating Branches

We consider now $\Lambda: R^+ \times H_r(D) \to L_r^2(D)$ defined by $\Lambda(\lambda, u) = \nabla \cdot [a^2(\|x\|) \nabla u] + \lambda f(u)$, where $H_r(D)$ stands for the radially symmetric functions of $H(D) = H_0^1(D) \cap H^2(D)$.

Lemma 3.1. 3.1.i. *C_k can be continued for all $\lambda > \lambda_k$.*

3.1.ii. *For any $i, j \geqslant 0$, $i \neq j$, we have $C_i \cap C_j = \phi$, in the sense that for any λ, $\lambda > \max\{\lambda_i, \lambda_j\}$, there are exactly two equilibria with i zeros in $(0, 1)$ and two equilibria with j zeros in $(0, 1)$. Moreover for any $k \geqslant 0$*

$$[C_k - (\lambda_k, 0)] \cap [(\lambda, 0), \lambda > \lambda_k, 0 \in H_r(D)] = \phi.$$

Proof. It follows from conditions (1.i), (1.ii), and (1.iii) that for any $\varepsilon > 0$ there is a constant c_ε such that $uf(u) \leqslant \varepsilon u^2 + c_\varepsilon$ on $\mathbb{R}$. This growth condition on f yields an a priori bound for the solutions of $\Lambda(\lambda, u) = 0$ with $u = 0$ on ∂D and λ in a compact set.

Indeed if u is a solution of the problem referred to above and $m = \min\{a^2(x), x \in \bar{D}\}$ then using the growth restriction on f, Poincaré inequality, the fact that $m > 0$, and Green's first identity we conclude that

$$m \int_D |\nabla u|^2 \, dx \leqslant \int_D a^2(\|x\|) \, |\nabla u|^2 \, dx$$

$$= \lambda \int_D uf(u) \, dx \leqslant \lambda \varepsilon K_1 \int_D |\nabla u|^2 \, dx + \lambda c_\varepsilon K_2,$$

where K_1 and K_2 are positive constants.

The following computation was used,

$$\int_D \nabla \cdot [a^2(r) \nabla u] u \, dx$$

$$= \int_D \nabla a^2(r) \cdot \nabla u \cdot u \, dx + \int_D a^2(r) u \, \Delta u \, dx$$

$$= -\int_D \nabla a^2(r) \cdot u \cdot \nabla u \, dx - \int_D \nabla(a^2(r) u) \nabla u \, dx + \int_{\partial D} a^2(r) u \frac{\partial u}{\partial v} \, ds$$

$$= -\int_D a^2(r) \, |\nabla u|^2,$$

where $r = \|x\|$.

Therefore $\int_D |\nabla u|^2 \leqslant \lambda c_\varepsilon K_2 / (m - \lambda \varepsilon K_1)$ and by choosing ε small enough and for λ in a compact set we conclude that $\|\nabla u\|_{L^2} \leqslant K_3$, K_3: const

Now let C_α, $\alpha \geqslant 0$, be any fixed bifurcating branch and let $\{\lambda_n\}_{n \in N}$ be an increasing sequence of real numbers such that $\lambda_n \to \lambda$ and $\lambda_n > \lambda_\alpha$, for any $n \in N$.

Let ϕ_n be the corresponding sequence of equilibrium solutions of (1.2), i.e., $\Lambda(\lambda_n, \phi_n) = 0$, with $\phi_n \in H_r(D)$.

It follows from the a priori boundedness of solutions that $\{\phi_n\}_{n \in N}$ is a bounded sequence in $H_0^1(D)$ and as a consequence there is a subsequence $\{\phi_{n,j}\}$ such that $\phi_{n,j} \to^{\omega} \phi$ in $H_0^1(D)$ and $\phi_{n,j} \to \phi$ in $L^2(D)$. Therefore $\lambda_{n,j} f(\phi_{n,j}) \to \lambda f(\phi)$ in $L^2(D)$.

Let us denote $\Gamma = -\operatorname{div}[a^2(\|x\|)\nabla \cdot]$ where Γ: $H_r^2(D) \to L^2(D)$. It is well known that as such, Γ has a compact resolvent and zero is in the resolvent set of Γ. Therefore

$$\Gamma^{-1}[\lambda_{n,j} f(\phi_{,j})] \to \Gamma^{-1}[\lambda f(\phi)] \qquad \text{in } H_r(D).$$

But $\phi_{n,j} = \Gamma^{-1}[\lambda_{n,j} f(\phi_{n,j})]$ and therefore $\Gamma(\phi) = \lambda f(\phi)$, i.e., $\operatorname{div}[a^2(\|x\|)\nabla\phi] + \lambda f(\phi) = 0$ in $L^2(D)$, with $\phi \in H_r(D)$.

Finally the claim about the extension on the bifurcating branch C_k, $k \geqslant 0$, follows from an application of the implicit function theorem. In order to do that we need to show that if $\phi_k \in C_k$ and $\lambda > \lambda_k$ then zero is not in the spectrum of the linearized operator $\Lambda_u(\lambda, \phi_k) = \nabla[a^2(\|x\|)\nabla u] + \lambda f'(\phi_k) u$ acting from

$$H_r(D) \qquad \text{into} \qquad L_r^2(D).$$

This will be shown in the next section.

4. Nondegeneracy of the Bifurcating Branches

The next theorem along with an application of the implicit function theorem shows that C_k, $k \geqslant 0$, is monotone in λ and presents no secondary bifurcations.

In working toward this goal the relevant point to be proved is that any $\phi_k \in C_k$, $k \geqslant 0$, is a nondegenerate equilibrium solution of (1.4) in the sense that zero is not in the spectrum of the linearized operator around ϕ_k.

Theorem 4.1. *Suppose that f satisfies* (1.i), (1.ii), (1.iii), *that the diffusion function $a^2(r)$ satisfies $r^2 a_{rr} + (N-1)\, r a_r \leqslant (N-1) a$, and that ϕ is an equilibrium solution of* (1.4). *Then $\mu = 0$ is not an eigenvalue of the following boundary value problem*:

$$(a^2(r)\,\psi_r)_r + (N-1)\,a^2(r)\,\frac{\psi_r}{r} + \lambda f'(\phi)\psi = \mu\psi, \qquad 0 < r < 1 \tag{4.1}$$

$$\psi_r(0) = \psi(1) = 0. \tag{4.2}$$

Proof. The idea of the proof is to suppose $\mu=0$ and then to show that there can be no nontrivial solution of (4.1), (4.2). More specifically we prove that if ψ is a nontrivial solution of (4.1) with $\psi_r(0)=0$ then necessarily $\psi(1)\neq 0$. We analyse the behavior of ψ for the equilibria ϕ_k's on each branch C_k, $k\geqslant 0$, in the following lemmas.

LEMMA 4.1. *If* $\phi_0 \in C_0$ *then Theorem* 4.1 *holds*

Proof. We render the proof only for the case $\phi_0\in C_0$, $\phi_0>0$, on $[0, 1)$. The case $\phi_0<0$ can be dealt with in similar manner.

The equality

$$\phi_{0,r}(r) = -\frac{\lambda}{r^{N-1}a^2(r)}\int_0^r s^{N-1}f(\phi_0(s))\,ds$$

can be obtained from (1.2) and shows that $\phi_{0,r}<0$ on $(0, 1]$.

Let ψ be a solution of Eq. (4.1), with $\mu=0$. We can assume $\psi(0)>0$. Suppose now, by contradiction, that there is an s_0, $0<s_0\leqslant 1$, such that $\psi>0$ on $[0, s_0)$ and $\psi(s_0)=0$. There are two cases to consider.

Case 1. If $0<s_0<1$, by setting $\xi=\psi/f(\phi_0)$ on $[0, s_0]$ we obtain after some computations,

$$\xi_{rr}+\left[\frac{(N-1)}{r}+\frac{(a^2)_r}{a^2}+\frac{2\phi_{0,r}f'(\phi_0)}{f(\phi_0)}\right]\xi_r$$
$$+\phi_{0,r}^2\frac{f''(\phi_0)}{f(\phi_0)}\xi=0,$$

$0<r<r_0$, and $\xi_r(0)=0$, $\xi(s_0)=0$, $\xi(0)>0$.

The coefficient of ξ in the above equation is negative so that a maximum principle assures us that ξ reaches its maximum at $r=0$ and $\xi_r(0)<0$. But this is a contradiction since $\xi_r(0)=0$. Therefore the only possibility is $\psi\equiv 0$ on $[0, s_0]$.

Case 2. Now suppose $s_0=1$ so that $\psi(1)=0$. Hence $\xi(0)>0$ and $\xi_r(0)=0$.

In order to compute $\xi_r(1)$ we use L'Hôpital's rule twice and the relations

$$\phi_{0,rr}(1) = -\left[2\frac{a_r(1)}{a(1)}+(N-1)\right]\phi_{0,r}(1)$$

$$\psi_{rr}(1) = -\left[2\frac{a_r(1)}{a(1)}+(N-1)\right]\psi_r(1),$$

to obtain that

$$\lim_{r \to 1} \xi_r(r) = 0.$$

Noting that the function

$$n(r) = \frac{(N-1)}{r} + \frac{(a^2(r))_r}{a^2(r)} + \frac{2\phi_{0,r}(r)\, f'(\phi_0(r))}{f(\phi_0(r))} + r\, \frac{f''(\phi_0(r))}{f(\phi_0(r))}\, \phi_{0,r}^2(r)$$

is bounded from below at $r=0$ and

$$m(r) = \frac{(N-1)}{r} + \frac{(a^2(r))_r}{a^2(r)} + \frac{2\phi_{0,r}(r)\, f'(\phi_0(r))}{f(\phi_0(r))} - \frac{(1-r)\, f''(\phi_0(r))\, \phi_{0,r}^2(r)}{f(\phi_0(r))}$$

is bounded from above at $r=1$, a maximum principle assures us that ξ reaches its maximum on $[0, 1]$ at one of the boundary points $r=0$, $r=1$ and that they have a nonzero derivative there. This is a contradiction since $\xi_r(0)=0$ and $\xi_r(1)=0$. Therefore $\psi \equiv 0$ on $[0, 1]$.

Summing up, for an equilibrium $\phi_0 \in C_0$, there cannot be any nontrivial solution ψ satisfying simultaneously (4.1) and (4.2) since we have just proved that the condition $\psi(1)=0$ fails to hold.

LEMMA 4.2. *If $\phi_1 \in C_1$ then Theorem* 4.1 *holds.*

Proof. Suppose ψ is a solution of (4.1), with $\mu=0$. It can be supposed that $\psi(0)>0$.

As a consequence there are points r_0 and r_1, $0<r_0<r_1<1$, such that $\phi_1(r_0)=0$ and $\phi_{1,r}(r_1)=0$.

By replacing ϕ_0 by ϕ_1 in Lemma 4.2 we conclude that $\psi>0$ on $[0, r_0]$, where ψ is a solution of (4.1), with $\mu=0$. It is our aim now to determine the number of zeros of ψ in $[r_0, 1]$ and their location. Any attempt to compare ψ with ϕ_1 now, as in the previous case, proves to be fruitless since $\xi = \psi/f(\phi_1)$ would blow up at $r=r_0$.

Therefore we compare ψ with $\phi_{1,r}$ as follows. Since ϕ_1 satisfies (1.2), taking derivatives of both sides of the equation, with $u=\phi_1$, we obtain:

$$\phi_{1,rrr} + 2\left(\frac{a_r}{a}\right)_r \phi_{1,r} + \frac{2a_r}{a}\,\phi_{1,rr} + (N-1)\,\frac{(r\phi_{1,rr} - \phi_{1,r})}{r^2} + \frac{f'(\phi_1)\,\phi_{1,r}a^2 - 2aa_r f(\phi_1)}{a^4}\,\lambda = 0 \qquad \text{for } 0<r<1.$$

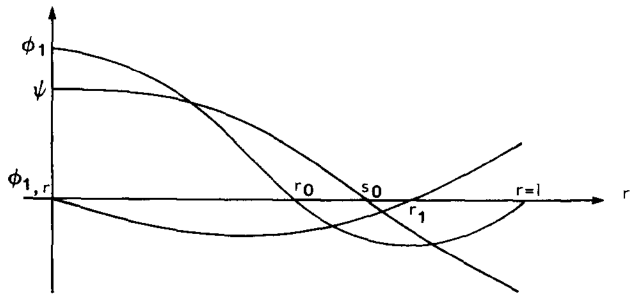

FIGURE 1

Next we suppose $\psi > 0$ or $[r_0, r_1]$ (see Fig. 1) and set $\xi = a\phi_{1,r}/\psi$ (note that in the original Eq. (1.4) the diffusion coefficient was $a^2(r)$). Using the above equality and boundary conditions we conclude after some computations that on $[0, r_1]$, ξ satisfies

$$\xi_{rr} + \left[2\,\frac{\psi_r}{\psi} + 2\,\frac{a_r}{a} + \frac{(N-1)}{r}\right]\xi_r + \left[\frac{a_{rr}}{a} + \frac{(N-1)\,a_r}{r\,a} - \frac{(N-1)}{r^2}\right]\xi = 0$$

$$\xi(0) = 0, \qquad \xi(r_1) = 0.$$

Note that the coefficients of ξ_r and ξ in the above equation are both bounded on every closed subinterval of $(0, r_1)$, while the latter one is non-positive by assumption. An application of the maximum principle to the above equation yields $\xi = 0$ on $[0, r_1]$ and hence $\phi_1 \equiv 0$ on $[0, r_1]$, which is a contradiction. So there is an s_0, $r_0 < s_0 < r_1$, such that $\psi(s_0) = 0$.

The next step is to prove that ψ has no zero on $(s_0, 1]$. To this end we suppose that there is an s_1, $s_0 < s_1 \leqslant 1$, such that $\psi(s_1) = 0$.

There are two cases to the analysed.

Case 1. If $s_0 < s_1 < 1$ then set $\xi = \psi/f(\phi_1)$ on $[s_0, s_1]$ and proceed just as in Case 1 of Lemma 4.1 with ψ_0 replaced by ϕ_1 and boundary conditions $\xi(s_0) = \xi(s_1) = 0$.

Again a maximum principle argument implies $\psi \equiv 0$ on $[s_0, s_1]$.

Case 2. If $s_1 = 1$ we proceed as in Case 1 above and argue as in Lemma 4.1, Case 2, to conclude that $\phi_1 \equiv 0$ on $[s_0, 1]$, which is a contradiction.

The case $\phi_1(0) < 0$ can be treated similarly.

Therefore if $\phi_1 \in C_1$ then necessarily $\psi(1) \neq 0$ and the only solution of (4.1), (4.2), with $\mu = 0$, is $\psi \equiv 0$.

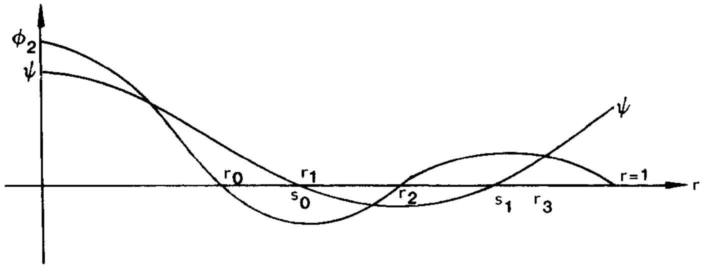

FIGURE 2

We now take $\phi_2 \in C_2$ just to illustrate how the foregoing lemmas can be used to prove the other cases.

If $\phi_2 \in C_2$, then there are r_0, r_1, r_2, r_3, and s_0 satisfying $0 < r_0 < s_0 < r_1 < r_2 < r_3 < 1$, $\phi_2(r_0) = \phi_2(r_2) = \phi_2(1) = 0$, $\phi_{2,r}(0) = \phi_{2,r}(r_1) = \phi_{2,r}(r_3) = 0$, and $\psi(s_0) = 0$ (see Fig. 2).

The behavior of ψ on $[0, r_2]$ can be obtained by using Lemma 4.1 on $[0, r_0]$ and then Lemma 4.2 on $[0, r_1]$.

We proceed as in Lemma 4.2 on $[r_0, r_1]$ to obtain s_1, $r_2 < s_1 < r_3$, such that $\psi(s_1) = 0$ and to prove that ψ does not vanish on $(s_1, 1]$, that is $\psi(1) \neq 0$.

An induction argument now shows that Theorem 4.1 holds for $\phi_n \in C_n$, $n \geqslant 0$.

If ϕ is a nontrivial equilibrium solution of (1.4) then ϕ has a finite number of simple zeros in $0 < r < 1$. This is so because the existence of a nonsimple zero would imply by the uniqueness of the solution that $\phi \equiv 0$ on $[0, 1]$. So this case falls in one of those referred to above.

The implicit function theorem can now be applied to show that each C_n, $n \geqslant 0$, is monotone in λ and does not present secondary bifurcation. Moreover, there can be no bifurcating branch starting off the set $\{C_n, n \geqslant 0\} \cup \{(\lambda, 0)/\lambda \in \mathbb{R}^+, 0 \in H(D)\}$.

5. STABILITY OF THE EQUILIBRIA

Throughout this section (5_D) will denote the equation

$$\frac{\partial u}{\partial t} = (a^2(r) u_r)_r + (N-1)\, a^2(r)\, \frac{u_r}{r} + \lambda f(u), \qquad 0 < r < 1,$$

followed by boundary conditions $u_r(0) = 0$ and $u(1) = 0$ and (5_N) will denote the same equation with boundary conditions $u_r(0) = 0$, $u_r(1) = 0$.

The study of the stability of the equilibria of problem (5_D) and (5_N)

above can be done via comparison techniques by establishing criteria which determine the equilibrium stability based on its number of zeros. The case $a^2(\|x\|) = \text{const}$ follows from a result by Casten and Holland [13], in the case of Neumann boundary conditions.

We first establish stability criteria which hold for the equilibrium solutions of problems (5_D) and (5_N) above and do not depend explicitly on the diffusion and reaction functions $a^2(\|x\|)$ and f, respectively. Then by imposing conditions, which were assumed in the previous sections, we prove our claims.

In what follows stability is meant in the sense of Lyapunov in the Sobolev space $H_0^1(0, 1)$ for problem (5_D) and in $H^1(0, 1)$ for problem (5_N).

The relationship between the concept of stability of an equilibrium solution of (5_D) $[(5_N)]$ in the sense of Lyapunov and the sign of the largest eigenvalue of the linearized operator around the equilibrium solution will be used hereafter without further comments.

The following stability criteria will be useful.

LEMMA 5.1. *Suppose that ϕ is an equilibrium solution of (5_D) and that $v \in C^2(0, 1)$ satisfies:*

$$L(v) = (a^2 v_r)_r + (N-1)\, a^2 \frac{v_r}{r} + \lambda f'(\phi) v \leqslant 0, \qquad 0 < r < 1$$

$$v(0) > 0, \qquad v_r(0) = 0.$$

If $v(r) > 0$ on $(0, 1]$, then ϕ is a stable equilibrium solutions of (5_D).

LEMMA 5.2. *Suppose that ϕ is an equilibrium solution of (5_D) $[(5_N)]$ and that $v \in C^2(0, 1)$ satisfies:*

$$L(v) = (a^2 v_r)_r + (N-1)\, a^2 \frac{v_r}{r} + \lambda f'(\phi) v \geqslant 0, \text{ as long as } v \text{ stays}$$

positive on $(0, 1]$ and $v(0) = 0$.

If $v(r)$ vanishes somewhere in $(0, 1)$ $[(0, 1]]$ then ϕ is an unstable equilibrium solution of (5_D) $[(5_N)]$.

In order to prove Lemma 5.2 the following Lemma will be needed.

LEMMA 5.3. *Suppose that ϕ, ψ, and ρ are C^2-functions satisfying $\rho(0) = \rho(r_1) = 0$, $\psi(0) \neq 0$, $\rho > 0$ on $(0, r_1)$, and*

$$(a^2\rho_r)_r + (N-1)\,a^2\,\frac{\rho_r}{r} + \lambda f'(\phi)\rho \geqslant 0 > (a^2\psi_r)_r$$
$$+ (N-1)\,a^2\,\frac{\psi_r}{r} + \lambda f'(\phi)\psi \qquad \textit{on } (0, r_1).$$

Then ψ *must vanish in* $(0, r_1]$.

Proof. Let us suppose by contradiction that $\psi > 0$ on $(0, r_1]$. Then by setting $\zeta = \rho/\psi$, we conclude that

$$(r^{N-1}a^2\psi^2\zeta_r)_r > 0 \qquad \text{on } (0, r_1].$$

Integrating this inequality from ε to r, where $0 < \varepsilon < r \leqslant r_1$, it follows that

$$r^{N-1}a^2\psi^2\zeta_r > \varepsilon^{N-1}a^2(\varepsilon)\,\psi^2(\varepsilon)\,\zeta_r(\varepsilon) \qquad \text{on } (0, r_1].$$

Moreover,

$$\lim_{\varepsilon\to 0} \zeta_r(\varepsilon) = \lim_{\varepsilon\to 0} \frac{\rho_r(\varepsilon)\,\psi(\varepsilon) - \rho(\varepsilon)\,\psi_r(\varepsilon)}{\psi^2(\varepsilon)} = \frac{\rho_r(0)}{\psi(0)},$$

which is finite. Therefore by taking the limit as $\varepsilon \to 0$, the above inequality yields $r^{n-1}a^2(r)\,\psi^2(r)\,\zeta_r \geqslant 0$ on $(0, r_1]$.

This implies $\zeta_r \geqslant 0$ on $(0, r_1]$, which is a contradiction since $\zeta(0) = \zeta(r_1) = 0$ and ψ must vanish in $(0, r_1]$.

Proof of Lemma 5.2. Suppose that ϕ is an equilibrium solution of (5_D) and that ψ_0 is an eigenfunction associated with the largest eigenvalue μ_0 of the following problem:

$$(a^2\psi_r)_r + (N-1)\,a^2\,\frac{\psi_r}{r} + \lambda f'(\phi)\psi = \mu\psi, \qquad 0 < r < 1$$
$$\psi_r(0) = 0, \qquad \psi(1) = 0.$$

It is well known that μ_0 is a simple eigenvalue and that ψ_0 has no zeros in $(0, 1)$.

Since by hypothesis v vanishes somewhere in $(0, 1)$, let r_1 stand for the first zero of v in $(0, 1)$. Let us suppose by contradiction that ϕ is stable, that is, $\mu_0 < 0$ and take $\psi_0 > 0$ on $[0, 1)$. Then

$$(a^2v_r)_r + (N-1)\,a^2\,\frac{v_r}{r} + \lambda f'(\phi)v \geqslant 0 \geqslant \mu_0\psi_0 = (a^2\psi_{o,r})_r$$
$$+ (N-1)\,a^2\,\frac{\psi_{o,r}}{r} + \lambda f'(\phi)\,\psi_0 \qquad \text{on } [0, r_1)$$

and

$$v(0) = v(r_1) = 0.$$

Lemma 5.3 implies that ψ_0 vanishes somewhere in $(0, r_1]$. But since $\psi_0 > 0$ on $(0, 1)$ we conclude μ_0 satisfies $\mu_0 > 0$. Therefore ϕ must be unstable.

Case (5_N) can be proved in a similar manner by allowing now $0 < r_1 \leqslant 1$.

Proof of Lemma 5.1. As in the proof of Lemma 5.2 let ψ_0 be an eigenfunction associated with the largest eigenvalue μ_0 of the corresponding eigenvalue problem.

Let us suppose by contradiction that ϕ is unstable, that is, $\mu_0 > 0$.

As a consequence we have,

$$L(v) \leqslant 0 < \mu_0 \psi_0 = L(\psi_0), \qquad \text{for } 0 < r < 1.$$

Moreover $v > 0$ on $[0, 1)$, $v_r(0) = 0$, $v(1) > 0$, and $\psi_0(1) = 0$, $\psi_{o,r}(1) < 0$. The above inequality can be written in the following form:

$$\begin{aligned}&(a^2 r^{N-1} v_r)_r + r^{N-1} \lambda f'(\phi) v \\ &\quad \leqslant 0 < (a^2 r^{N-1} \psi_{o,r})_r + r^{N-1} \lambda f'(\phi)\, \psi_0, \qquad 0 < r < 1.\end{aligned}$$

Using the fact that $\psi_0 > 0$ and $v > 0$ in $[0, 1)$ we obtain $\psi_0 (a^2 r^{N-1} v_r)_r \leqslant v(a^2 r^{N-1} \psi_{o,r})_r$, for $0 < r < 1$, which in turn can be integrated to yield $\psi_0(1)\, v_r(1) \leqslant v(1)\, \psi_{o,r}(1)$, that is, $0 \leqslant \psi_{o,r}(1)$, in contradiction with the conditions above. Therefore $\mu_0 < 0$ and ϕ is a stable equilibrium solution of (5_D).

THEOREM 5.1. *If the diffusion function $a^2(r)$ satisfies the inequality $r^2 a_{rr} + (N-1)\, r a_r \leqslant (N-1) a$, for $0 < r < 1$, then every nonconstant equilibrium solution ϕ of (5_D), such that ϕ_r vanishes somewhere in $(0, 1)$, is unstable.*

Proof. Let ϕ be a nonconstant equilibrium solution of (5_D) such that ϕ_r vanishes somewhere in $(0, 1)$. It may be supposed that r_0 is the first zero of ϕ_r in $(0, 1)$ and that $\phi_r > 0$ in $(0, r_0)$. It is our goal now to compute $L(a\phi_r)$, where

$$L(a\phi_r) = [a^2 (a\phi_r)_r]_r + (N-1)\, a^2 \frac{(a\phi_r)_r}{r} + \lambda f'(\phi)\, a\phi_r,$$

$$\text{for } r \in (0, r_0).$$

Since ϕ is an equilibrium solution of (5_D), it satisfies

$$(a^2\phi_r)_{rr} + (N-1)(a^2)_r \frac{\phi_r}{r} + (N-1)\, a^2 \left(\frac{\phi_r}{r}\right)_r$$
$$+ \lambda f'(\phi)\, \phi_r = 0, \qquad 0 < r < 1.$$

This equation can be used in the manipulation of the expression $L(a\phi_r)$ in order to eliminate the term involving $f'(\phi)$.

After some computations we conclude that

$$L(a\phi_r) = \left[(N-1)\, \frac{a^2}{r^2} - aa_{rr} - (N-1)\, \frac{aa_r}{r} \right] (a\phi_r).$$

From our hypothesis it follows that $L(a\phi_r) \geqslant 0$ in $(0, r_0)$. Let us call $v(r) = a(r)\, \phi_r(r)$.

This function v satisfies $L(v) \geqslant 0$, $v > 0$, on $(0, r_0)$, $v(0) = 0$, and v vanishes at a point in $(0, 1)$.

Now we resort to Lemma 5.2 to conclude that ϕ is an unstable equilibrium solution of (5_D).

THEOREM 5.2. *If the diffusion function $a^2(r)$ satisfies $r^2 a_{rr} + (N-1)\, ra_r \leqslant (N-1)a$ on $(0, 1)$ then every nonconstant equilibrium solution of (5_N) is unstable.*

Proof. Let us suppose that ϕ is a nonconstant equilibrium solution of (5_N). Therefore there must exist r_0, $0 < r_0 \leqslant 1$, such that $\phi_r(r_0) = 0$ and $\phi_r \neq 0$ on $(0, r_0)$. We can suppose without loss of generality that $\phi_r > 0$ on $(0, r_0)$.

As before, by making $v = a\phi_r$ on $[0, r_0]$, we obtain a function v satisfying $L(v) \geqslant 0$, $v(0) = v(r) = 0$ with $0 < r_0 \leqslant 1$ and Lemma 5.2 implies that ϕ is an unstable equilibrium solution of (5_N).

THEOREM 5.3. *Suppose that the reaction function f is a C^2-function satisfying*

(i) $f(0) = 0$, $f'(0) > 0$,

(ii) $\operatorname{sgn} f''(u) = -\operatorname{sgn} u$, *for all* $u \in R$, $u \neq 0$.

Then, regardless of the diffusion function a^2 every positive or negative equilibrium solution of (5_D) on $(0, 1)$ is stable.

Proof. For the sake of simplicity in notation, in this case, we delete the square of the diffusion term $a^2(r)$, since the derived conditions will not depend on it and render the proof only for the case of a positive equilibrium solution.

Let $v \in C^2(0, 1)$ satisfy

$$L(v) = (av_r)_r + (N-1)a\,\frac{v_r}{r} + \lambda f'(\phi)v = 0, \qquad 0 < r < 1$$

and $v_r(0) = 0$ where ϕ is a nonconstant positive equilibrium solution of (5_D).

Suppose that v vanishes somewhere in $(0, 1]$, say, at r_0.

Consider first the case $0 < r_0 < 1$. Then by setting $\zeta = v/f(\phi)$ on $[0, r_0]$, we obtain after a computation:

$$\zeta_{rr} + \left[\frac{(N-1)}{r} + \frac{a_r}{a} + \frac{2\phi_r f'(\phi)}{f(\phi)}\right]\zeta_r$$

$$+ \phi_r^2 \frac{f''(\phi)}{f(\phi)}\zeta = 0, \qquad 0 < r < r_0$$

$$\zeta_r(0) = 0, \qquad \zeta(r_0) = 0$$

By virtue of our hypothesis the coefficient of ζ in the above equation is negative, so that a maximum principle assures us that ζ reaches its maximum either at $r = 0$ or at $r = r_0$. This is not possible since $\zeta > 0$ on $(0, r_0)$ and $\zeta_r(0) = \zeta(r_0) = 0$.

Therefore only the case $r_0 = 1$ remains to be analysed.

In working toward this goal, the relevant point to be proved is that

$$\lim_{s \to 1} \zeta_r(s) = \frac{-f''(0)\, v_r(1)}{2[f'(0)]^2} = 0.$$

This can be accomplished by using L'Hôpital's rule twice and the relations $f''(0) = 0$, $v_r(1)\,\phi_{rr}(1) = v_{rr}(1)\,\phi_r(1)$, which can be deduced from our hypothesis.

Then ζ would still be governed by the equation above on $(0, 1)$, with boundary conditions now replaced by $\zeta_r(0) = 0$, $\zeta(1) > 0$, and $\zeta > 0$ on $(0, 1)$.

A maximum principle assures us that $\zeta_r(1) < 0$, which is a contradiction.

Hence v cannot vanish in $(0, 1]$ and by resorting to Lemma 5.1 we conclude that ϕ is a stable equilibrium solution of (5_D).

Under the hypothesis of Theorem 5.3, a nonconstant positive equilibrium solution ϕ of (5_D) is indeed a decreasing function on $(0, 1]$. This can be seen from the relation

$$\phi_r(r) = -\frac{\lambda}{r^{N-1}a(r)} \int_0^r s^{N-1} f(\phi(s))\, ds$$

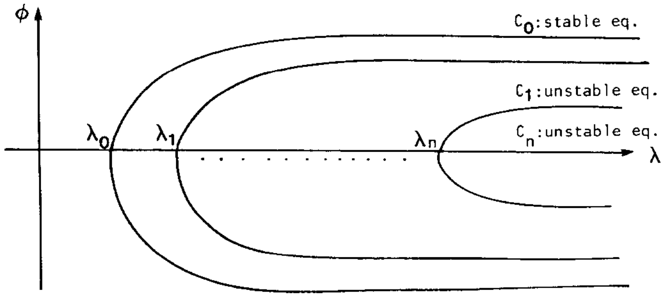

FIGURE 3

and a maximum principle argument in case $f(u)$ vanishes somewhere for $u > 0$.

In a similar manner it can be seen that if f satisfies the hypothesis of Theorem 5.3, a nonconstant negative equilibrium solution of (5_D) must be an increasing function on $(0, 1]$.

Remark 1. Under the assumptions of Theorem 5.3 the only possible nonvanishing equilibrium solutions of (5_N) are the zeros of f (if they exist) and in this case they are stable since at a zero p of f we have $f'(p) < 0$ if $p \neq 0$.

THEOREM 5.4. *If the function $a^2(r)$ satisfies $r^2 a_{rr} + (N-1)\, r a_r \leqslant (N-1) a$, then the equilibrium solutions of* (1.4) *in C_k, $k \geqslant 0$, are stable for $k = 0$ and unstable for $k \geqslant 1$.*

Therefore summing up the results of this section, we conclude that the bifurcation diagram for the equilibrium solutions of (1.4), *shown along with the equilibria stability, is as shown in Fig.* 3.

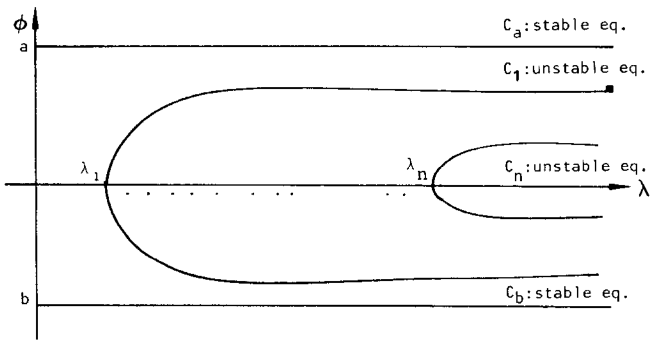

FIGURE 4

If the function f has two zeros a and b, $a<0<b$, then the same diagram for the same equation with a Neumann boundary condition is as shown in Fig. 4.

Remark 2. It is worthwhile to note that as far as nondegeneracy and stability of the bifurcating branches are concerned, the condition on the diffusion function, namely $r^2 a_{rr} + (N-1) r a_r \leqslant (N-1) a$, was not required for the equilibria on the branch C_0. In other words, if the reaction function f satisfies conditions (i) and (ii) of Theorem 5.3, then the branch C_0 (of those equilibria which do not change sign in D) is monotone in λ, does not present secondary bifurcation, and is stable regardless of the function $a^2(r)$.

Therefore in order that the branch C_0 loses its monotonicity in λ the function f is the parameter to be dealt with and f should be allowed to have more oscillation.

Remark 3. As previously remarked, resorting to Remark 2 and the results of Section 5 we conclude that Theorem 5.1 and Theorem 5.2 extend results of Hale and Chipot [3] and Yanagida [2] to radially symmetric equilibria of (1.1) in a ball.

Remark 4. The present work has consequences in the study of the asymptotic behavior of radially symmetric solutions to problem (1.1). Namely, resorting to well-known results of dynamical systems it can be concluded that, for a fixed λ, "almost all" radially symmetric solutions to (1.1) converge to the two equilibria in the branch C_0, one of which is positive on D and the other which is negative on D.

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